Poisson-Jacobi reduction of homogeneous tensors *

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Abstract

The notion of homogeneous tensors is discussed. We show that there is a one-to-one correspondence between multivector fields on a manifold M, homogeneous with respect to a vector field Δ on M, and first-order polydifferential operators on a closed submanifold N of codimension 1 such that Δ is transversal to N. This correspondence relates the Schouten-Nijenhuis bracket of multivector fields on M to the Schouten-Jacobi bracket of first-order polydifferential operators on N and generalizes the Poissonization of Jacobi manifolds. Actually, it can be viewed as a super-Poissonization. This procedure of passing from a homogeneous multivector field to a first-order polydifferential operator can be also understood as a sort of reduction; in the standard case – a half of a Poisson reduction. A dual version of the above correspondence yields in particular the correspondence between Δ-homogeneous symplectic structures on M and contact structures on N.

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1 Introduction

As it has been observed in [KoS], a Lie algebroid structure on a vector bundle E can be identified with a Gerstenhaber algebra structure on the exterior algebra of multisections of E, $Sec(\wedge E)$, which is just a graded Poisson bracket (Schouten bracket) on $Sec(\wedge E)$ of degree -1, that is, the Schouten bracket is graded commutative, satisfies the graded Jacobi identity and the graded Leibniz rule.

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In the particular case of the Lie algebroid structure on the tangent vector bundle of an arbitrary manifold M one obtains the Schouten-Nijenhuis bracket $[\![\cdot,\cdot]\!]_M$ on the space of multivectors on M.

For a graded commutative algebra with 1, a natural generalization of a graded Poisson bracket is a graded Jacobi bracket: we replace the graded Leibniz rule by that $\{a,\cdot\}$ is a first-order differential operator on $\mathcal{A}$, for every $a \in \mathcal{A}$ (cf. [GM2]).

Graded Jacobi brackets on $Sec(\wedge E)$ of degree -1 are called Schouten-Jacobi brackets. These brackets are in one-to-one correspondence with pairs (E,ϕ_0), where $\phi_0 \in Sec(E^*)$ is a 1-cocycle in the Lie algebroid cohomology of E. In this case, we said that (E,ϕ_0) is a generalized Lie algebroid (Jacobi algebroid) (see [GM1, IM2]).

A canonical example of a Jacobi algebroid is $(T^1M,(0,1))$ where $T^1M = TM \oplus \mathbb{R}$ is the Lie algebroid of first-order differential operators on the space of smooth functions on M, $C^\infty(M)$, with the bracket

$$[\![X \oplus f, Y \oplus g]\!]^1_M = [X,Y] \oplus (X(g) - Y(f)),$$

for $X \oplus f, Y \oplus g \in Sec(T^1M)$ (see [M, N]) and the 1-cocycle $\phi_0 = (0,1) \in \Omega^1(M) \oplus C^\infty(M)$.

It is well-known that a Poisson structure on a manifold M can be interpreted as a canonical structure for the Schouten-Nijenhuis bracket $[\![\cdot,\cdot]\!]_M$ of multivector fields on M, i.e., as an element $\Lambda \in Sec(\wedge^2 TM)$ satisfying the equation $[\![\Lambda,\Lambda]\!]_M = 0$. In similar way, a Jacobi structure is a canonical structure for the Jacobi bracket $[\![\cdot,\cdot]\!]^1_M$.

On the other hand, it is proved in [DLM] that if Λ is a homogeneous Poisson tensor with respect to a vector field Δ on the manifold M and N is a 1-codimensional closed submanifold of M such that Δ is transversal to N then Λ can be reduced to a Jacobi structure on N.

The main purpose of this paper is to give an explicit (local) correspondence between Δ-homogeneous multivector fields on M and first-order polydifferential (i.e. skew-symmetric multidifferential) operators on N. This correspondence relates the Schouten-Nijenhuis bracket of multivector fields on M to the Schouten-Jacobi bracket of first-order polydifferential operators on N. This is of course a generalization of [DLM] formulated in a structural way. It explains the role of homogeneity for certain reduction procedures, e.g. in passing from Poisson to Jacobi brackets (in mechanics: from symplectic form to a contact form). But our result can be applied in Nambu-Poisson geometry (cf. Corollary 3.13) or multisymplectic geometry and classical field theories as well.

The paper is organized as follows. In Section 2 we recall the notions of Schouten-Nijenhuis and Schouten-Jacobi brackets associated with any smooth manifold. In Section 3.1 we introduce the notion of Δ-homogeneous tensors on a homogeneous structure (M,Δ) (a pair where M is a manifold and Δ is a vector field on M).

Moreover, for a particular class of homogeneous structures (strict homogeneous structures), we will characterize the Δ-homogeneous contravariant k-tensors in terms of their corresponding k-ary brackets.

The main result of the paper is Theorem 3.11 of Section 3.2, which provides the one-to-one correspondence between homogeneous multivector fields and polydifferential operators we have already mentioned. This result is a generalization of the result of [DLM] cited above and it allows us also to relate homogeneous Nambu-Poisson tensors on M to Nambu-Jacobi tensors on N. These results are local. We obtain global results in the particular case of the Liouville vector field $\Delta = \Delta_E$ of a vector bundle $\tau : E \to M$. We called this correspondence a Poisson-Jacobi reduction, since it can be understood as a sort of reduction, a half of a Poisson reduction (cf. Remark 3.12, ii)).

Finally, we prove a dual version of Theorem 3.11. What we get is a one-to-one correspondence between homogeneous differential forms on M and elements of $Sec(\wedge(T^*N \oplus \mathbb{R}))$ represented by pairs (α^0, α^1), where α^0 is a k-form on N and α^1 is a $(k-1)$-form on N. This correspondence relates the de Rham differential on M with deformed Lie algebroid differential associated with the Schouten-Jacobi bracket $[\![\cdot,\cdot]\!]_M^1$ (see [IM2, GM1]).

Note that the Grassmann algebra $Sec(\wedge TM)$ can be viewed as the algebra of functions on the supermanifold ΠT^*M (the space of the cotangent bundle to M with reversed parity of fibers, cf. [AKSZ]) the Schouten-Nijenhuis bracket on $Sec(\wedge TM)$ represents the canonical (super) Poisson bracket on ΠT^*M. In this picture, the equation $[\![\Lambda,\Lambda]\!]_M = 0$ for a Poisson tensor Λ is just a particular case of the Master Equation in Batalin-Vilkovisky formalism. The algebraic structure of Batalin-Vilkovisky formalism in field theories (see [Ge]) have been recognized as a homologic vector field generating a Schouten-Nijenhuis-type bracket on the corresponding graded commutative algebra like the Schouten-Nijenhuis bracket (Gerstenhaber algebra) of a Lie algebroid [KoS, KS2]. The Schouten-Jacobi bracket can be regarded as a super-Jacobi bracket, so Theorem 3.11 can be understood as a super or fermionic version of the original result [DLM]. Note also that higher-order tensors represent higher-order operations on the ring of functions. Together with the Schouten-Nijenhuis or Schouten-Jacobi bracket, possibly for higher gradings, this can be a starting point for certain strongly homotopy algebras (cf. the paper [St] by J. Stasheff who realized that homotopy algebras appear in string field theory). A relation of some strongly homotopy algebras with Batalin-Vilkovisky formalism was discovered by B. Zwiebach and applied to string field theory [Zw]. Theorem 3.11 means that in homogeneous cases we can reduce the structure to the same super Lie bracket on a smaller manifold. The difference is that we deal not with derivations but with first-order differential operators. The structure of the associative product is deformed by this bracket isomorphism, so we get not a super Poisson but a super Jacobi bracket. On the level of differential forms this corresponds to a deformation of the de Rham differential of the type $d^1\mu = d\mu + \phi\wedge\mu$, where ϕ is a closed 1-form. This is exactly what was already considered by E. Witten [Wi] and used in studying of spectra of Laplace operators.

2 Graded Lie brackets

In this section we will recall several natural graded Lie brackets of tensor fields associated with any smooth manifold M. First of all, on the tangent bundle TM, we have a Lie algebroid bracket $[\cdot,\cdot]$ defined on the space $\mathfrak{X}(M)$ of vector fields – derivations of the algebra $C^\infty(M)$ of smooth functions on M.

If $A(M) = \oplus_{k\in\mathbb{Z}} A^k(M)$ is the space of multivector fields (i.e., $A^k(M) = Sec(\wedge^k TM)$) then we can define the Schouten-Nijenhuis bracket (see [Sc, Ni]) $[\![\cdot,\cdot]\!]_M : A^p(M)\times A^q(M) \to A^{p+q-1}(M)$ as the unique graded extension to $A(M)$ of the bracket $[\cdot,\cdot]$ of vector fields, such that:

i) $[\![X,f]\!]_M = X(f)$, for $X\in\mathfrak{X}(M)$ and $f\in C^\infty(M)$;

ii) $[\![P,Q]\!]_M = -(-1)^{(p-1)(q-1)}[\![Q,P]\!]_M$, for $P\in A^p(M)$, $Q\in A^q(M)$.

iii) $[\![P,Q\wedge R]\!]_M = [\![P,Q]\!]_M\wedge R + (-1)^{(p-1)q} Q\wedge[\![P,R]\!]_M$, for $P\in A^p(M)$, $Q\in A^q(M)$ and $R\in A^*(M)$;

iv) $(-1)^{(p-1)(r-1)}[\![[\![P,Q]\!]_M,R]\!]_M + (-1)^{(p-1)(q-1)}[\![[\![Q,R]\!]_M,P]\!]_M + (-1)^{(q-1)(r-1)}\ [\![[\![R,P]\!]_M,\ Q]\!]_M = 0$, for $P\in A^p(M)$, $Q\in A^q(M)$ and $R\in A^r(M)$.

On the other hand, if $\Omega(M) = \oplus_{k\in\mathbb{Z}}\Omega^k(M)$ is the space of differential forms (that is, $\Omega^k(M) = Sec(\wedge^k(T^*M))$) we can consider the usual differential $d_M : \Omega^p(M)\to\Omega^{p+1}(M)$ as the map characterized by the following properties:

(i) d_M is a $\mathbb{R}$-linear map.

(ii) $d_M(f)$ is the usual differential of f, for $f \in C^\infty(M)$.

(iii) $d_M(\alpha \wedge \beta) = d_M\alpha \wedge \beta + (-1)^p \alpha \wedge d_M\beta$, for $\alpha \in \Omega^p(M)$ and $\beta \in \Omega^q(M)$.

(iv) $d_M^2 = 0$, that is, d_M is a cohomology operator.

In a similar way, on the bundle of first-order differential operators on $C^\infty(M)$, $T^1M = TM \oplus \mathbb{R}$, there exists a Lie algebroid bracket given by

$$[\![X \oplus f, Y \oplus g]\!]^1_M = [X, Y] \oplus (X(g) - Y(f)), \tag{2.1}$$

for $X \oplus f, Y \oplus g \in Sec(T^1M)$ (see [M, N]).

The space $D^k(M) = Sec(\wedge^k(T^1M))$ of sections of the vector bundle $\wedge^k(T^1M) \to M$ can be identified with $A^k(M) \oplus A^{k-1}(M)$ in the following way. If $I_M = 0 \oplus 1_M \in Sec(T^1M)$ and $\phi_M \in Sec((T^1M)^*)$ is the "canonical closed 1-form" defined by $\phi_M(X \oplus f) = f$, then there exists an isomorphism between $D^k(M)$ and $A^k(M) \oplus A^{k-1}(M)$ given by the formula:

$$\begin{array}{rcl} D^k(M) = Sec(\wedge^k(T^1M)) & \to & A^k(M) \oplus A^{k-1}(M) \\ D & \mapsto & D^0 \oplus D^1 \cong D^0 + I_M \wedge D^1, \end{array}$$

where $D^1 = i_{\phi_M} D$ and $D^0 = D - I_M \wedge D^1$.

As for $A(M)$, we can define on $D(M) = \oplus_{k\in\mathbb{Z}} D^k(M)$ a canonical Schouten-Jacobi bracket $[\![\cdot,\cdot]\!]^1_M$: $D^k(M) \times D^r(M) \to D^{k+r-1}(M)$ (see [GM1, IM2])

$$\begin{aligned} [\![P^0 + I_M \wedge P^1, Q^0 + I_M \wedge Q^1]\!]^1_M \quad = \\ & [\![P^0, Q^0]\!]_M + (k-1)P^0 \wedge Q^1 + (-1)^k (r-1) P^1 \wedge Q^0 \\ & + I_M \wedge \Big([\![P^1, Q^0]\!]_M - (-1)^k [\![P^0, Q^1]\!]_M + (k-r) P^1 \wedge Q^1 \Big), \end{aligned} \tag{2.2}$$

for $P = P^0 + I_M \wedge P^1 \in D^k(M)$ and $Q = Q^0 + I_M \wedge Q^1 \in D^r(M)$. The bracket $[\![\cdot,\cdot]\!]^1_M$ is the unique graded bracket characterized by:

i) It extends the Lie bracket on $D^1(M)$ defined by (2.1);

ii) $[\![X \oplus f, g]\!]^1_M = X(g) + fg$, for $X \oplus f \in D^1(M)$ and $g \in C^\infty(M)$;

iii) $[\![D, E]\!]^1_M = -(-1)^{(p-1)(q-1)} [\![E, D]\!]_M$, for $D \in A^p(M)$, $E \in A^q(M)$.

iv) $[\![D, E \wedge F]\!]^1 = [\![D, E]\!]^1_M \wedge F + (-1)^{(p-1)q} E \wedge [\![D, F]\!]^1_M - (i_{\phi_M} D) \wedge E \wedge F$,

for $D \in D^p(M)$, $E \in D^q(M)$ and $F \in D^*(M)$;

v) $(-1)^{(p-1)(r-1)} [\![[\![D, E]\!]^1_M, F]\!]^1_M + (-1)^{(p-1)(q-1)} [\![[\![E, F]\!]^1_M, D]\!]^1_M + (-1)^{(q-1)(r-1)} [\![[\![F, D]\!]^1_M, E]\!]^1_M = 0$, for $D \in D^p(M)$, $E \in D^q(M)$ and $F \in D^r(M)$.

On the other hand, the space $\Theta^k(M) = Sec(\wedge^k(T^1M)^*)$ of sections of the vector bundle $\wedge^k(T^1M)^* \to M$ can be identified with $\Omega^k(M) \oplus \Omega^{k-1}(M)$. Actually, there exists an isomorphism between $\Theta^k(M)$ and $\Omega^k(M) \oplus \Omega^{k-1}(M)$ given by the formula

$$\begin{array}{rcl} \Theta^k(M) = Sec(\wedge^k(T^1M)^*) & \to & \Omega^k(M) \oplus \Omega^{k-1}(M) \\ \alpha & \to & \alpha^0 \oplus \alpha^1 \cong \alpha^0 + \phi_M \wedge \alpha^1 \end{array}$$

where

$$\alpha^1 = i_{I_M}\alpha, \quad \alpha^0 = \alpha - \phi_M \wedge \alpha^1.$$

In other words,

$$\alpha(X_1 \oplus f_1, \ldots, X_k \oplus f_k) \;=\; \alpha^0(X_1, \ldots, X_k) + \sum_{i=1}^{k}(-1)^{i+1} f_i \alpha^1(X_1, \ldots, \hat{X}_i, \ldots, X_k)$$

for $X_1 \oplus f_1, \ldots, X_k \oplus f_k \in Sec(T^1M)$.

As for $\Omega(M)$, we can define on $\Theta(M) = \oplus_{k\in\mathbb{Z}}\Theta^k(M)$ the Jacobi differential $d^1_M : \Theta^k(M) \to \Theta^{k+1}(M)$ as the map characterized by the following properties:

(i) d^1_M is a $\mathbb{R}$-linear map.

(ii) If $f \in C^\infty(M)$ and $j^1 f \in Sec((T^1M)^*)$ is the first jet prolongation of f then $d^1_M f = j^1 f$.

(iii) $d^1_M(\alpha \wedge \beta) = d^1_M\alpha \wedge \beta + (-1)^p \alpha \wedge d^1_M\beta - \phi_M \wedge \alpha \wedge \beta$, for $\alpha \in \Theta^p(M)$ and $\beta \in \Theta^q(M)$.

(iv) $(d^1_M)^2 = 0$, that is, d^1_M is a cohomology operator.

Under the isomorphism between $\Theta^k(M)$ and $\Omega^k(M) \oplus \Omega^{k-1}(M)$ the operator d^1_M is given by

$$d^1_M(\alpha^0, \alpha^1) = (d_M\alpha^0, -d_M\alpha^1 + \alpha^0),$$

for $(\alpha^0, \alpha^1) \in \Omega^k(M) \oplus \Omega^{k-1}(M) \cong \Theta^k(M)$.

To finish with this section, we recall that it is easy to identify $P \in A^k(M)$ (resp., $D = D^0 + I_M \wedge D^1 \in D^k(M)$) with a polyderivation $\{\cdot, \ldots, \cdot\}_P\colon C^\infty(M)\times \overset{k)}{\ldots} \times C^\infty(M) \to C^\infty(M)$ (resp., a first-order polydifferential operator $\{\cdot, \ldots, \cdot\}_D\colon C^\infty(M)\times \overset{k)}{\ldots} \times C^\infty(M) \to C^\infty(M)$) given by

$$\{f_1, \ldots, f_k\}_P = \langle P, df_1 \wedge \ldots \wedge df_k\rangle \tag{2.3}$$

(resp.,

$$\begin{aligned} \{f_1, \ldots, f_k\}_D \;&=\; \langle D, j^1 f_1 \wedge \ldots \wedge j^1 f_k\rangle = \{f_1, \ldots, f_k\}_{D^0} \\ &\quad + \sum_{i=1}^{k}(-1)^{i+1} f_i \{f_1, \ldots, \widehat{f_i}, \ldots, f_k\}_{D^1}) \end{aligned} \tag{2.4}$$

for all $f_1, \ldots, f_k \in C^\infty(M)$. Note that $(A(M), [\![\,,\,]\!]_M)$ is naturally embedded into $(D(M), [\![\,,\,]\!]^1_M)$. Actually, elements of $(A(M), [\![\,,\,]\!]_M)$ are just those $D \in (D(M), [\![\,,\,]\!]^1_M)$ for which $i_{\phi_M} D = 0$.

3 Homogeneous structures

3.1 Homogeneous tensors

In this Section we will consider a particular class of tensors related to a distinguished vector field on a manifold.

Let M be a differentiable manifold and let Δ be a vector field on M. The pair (M, Δ) will be called a *homogeneous structure*.

A function $f \in C^\infty(M)$ is Δ-*homogeneous of degree* n, $n \in \mathbb{R}$, if $\Delta(f) = n\, f$. The space of Δ-homogeneous functions of degree n will be denoted by $S^n_\Delta(M)$. Similarly, a tensor T is Δ-*homogeneous of degree* n if $\mathcal{L}_\Delta T = nT$. Here $\mathcal{L}$ denotes the Lie derivative. In particular, Δ itself is homogeneous of degree zero. As a result of properties of the Lie derivative we get the following properties of the introduced homogeneity gradation.

(i) The tensor product $T \otimes S$ of Δ-homogeneous tensors of degrees n and m respectively, is homogeneous of degree $n + m$.

(ii) The contraction of tensors of homogeneity degrees n and m is homogeneous of degree $n + m$.

(iii) The exterior derivative preserves the homogeneity degree of forms.

(iv) The Schouten-Nijenhuis bracket of multivector fields of homogeneity degrees n and m is homogeneous of degree $n + m$.

These properties justify our choice of the homogeneity gradation, which is compatible with the polynomial gradation introduced in [TU] and differs by a shift from homogeneity gradation of contravariant tensors in some other papers (e.g [Li]).

Example 3.1 *i)* The simplest example of a homogeneous structure is the pair $(N \times \mathbb{R}, \partial_s)$, where ∂_s is the canonical vector field on $\mathbb{R}$. $(N \times \mathbb{R}, \partial_s)$ will be called a *free homogeneous structure*. In this case,

$$S^n_\Delta(M) = \{f \in C^\infty(N \times \mathbb{R})\colon\, f(x,s) = \mathrm{e}^{ns} f_N(x),\ \text{with } f_N \in C^\infty(N),\, \forall\, (x,s) \in N \times \mathbb{R}\}.$$

ii) Let $M = N \times \mathbb{R}$ and $\Delta = s\partial_s$, s being the usual coordinate on $\mathbb{R}$. In this case

$$S^n_\Delta(M) = \{f \in C^\infty(N \times \mathbb{R})\colon\, f(x,s) = s^n f_N(x),\ \text{with } f_N \in C^\infty(N),\, \forall\, (x,s) \in N \times \mathbb{R}\}.$$

iii) If $M = \mathbb{R}$ and $\Delta = s^2\partial_s$, then $S^0_\Delta(M) = \mathbb{R}$ and $S^n_\Delta(M) = \{0\}$ for $n \neq 0$ because the differential equation $s^2 \frac{\partial f}{\partial s} = nf$ has no global smooth solutions on $\mathbb{R}$ for $n \neq 0$.

Using coordinates adapted to the vector field, one can easily prove the following result.

Proposition 3.2 *Let (M, Δ) be a homogeneous structure and N be a closed submanifold in M of codimension 1 such that Δ is transversal to N. Then, there is a tubular neighborhood U of N in M and a diffeomorphism of U onto $N \times \mathbb{R}$ which maps $\Delta_{|U}$ into ∂_s*

Let us introduce a particular class of homogeneous structures which will be important in the sequel.

Definition 3.3 *A homogeneous structure (M, Δ) is said to be* strict *if there is an open-dense subset $O \subset M$ such that for $x \in O$*

$$T^*_x M = \{df(x)\colon\, f \in S^1_\Delta(M)\}.$$

Example 3.4 *i)* It is almost trivial that free homogeneous structures are strict homogeneous.

ii) An example of a strict homogeneous structure with Δ vanishing on a submanifold is the following. Let $E \to M$ be a vector bundle (of rank > 0) over M and let $\Delta = \Delta_E$ be the *Liouville* vector field on E. Then, for $n \in \mathbb{Z}_+$, $S^n_\Delta(E)$ consists of smooth functions on E which are homogeneous polynomials of degree n along fibres. In particular, functions from $S^1_\Delta(E)$ are linear on fibres, hence generate T^*E over E_0, the bundle E with the zero-section removed.

Now, generalizing the situation for tensors, we will consider first-order polydifferential operators.

For a homogeneous structure (M, Δ), we say that $D \in D^k(M)$ is Δ*-homogeneous of degree* n if $[\![\Delta, D]\!]^1_M = nD$. For $P \in A^k(M)$ interpreted as an element of $D(M)$, it is Δ-homogeneous of degree n when $[\![\Delta, P]\!]_M = \mathcal{L}_\Delta P = nP$, i.e. the introduced gradation is compatible with the gradation for tensors. It is easy to see, using (2.2), that $P = P^0 + I_M \wedge P^1 \in D^k(M)$ is Δ-homogeneous of degree n if and only if $P^0 \in A^k(M)$ and $P^1 \in A^{k-1}(M)$ are Δ-homogeneous of degree n. In particular, the identity operator is homogeneous of degree zero.

Elements of $D^k(M)$ which are Δ-homogeneous of degree $1-k$ we will call simply Δ*-homogeneous*.

Proposition 3.5 *Suppose that* $D \in D^k(M)$ *is* Δ*-homogeneous of degree* n *and suppose that* $D' \in D^{k'}(M)$ *is* Δ*-homogeneous of degree* n'*. Then,*

i) $D \wedge D'$ *is* Δ*-homogeneous of degree* $n + n'$.

ii) $[\![D, D']\!]^1_M$ *is* Δ*-homogeneous of degree* $n + n'$.

Proof.- These properties are immediate consequences of properties of the Schouten-Jacobi bracket $[\![\cdot,\cdot]\!]^1_M$ (see Section 2) and the fact that $i_{\phi_M}\Delta = 0$. □

We can characterize homogeneous operators for strict homogeneous structures in terms of the corresponding k-ary brackets as follows.

Proposition 3.6 *Let* (M, Δ) *be a strict homogeneous structure. Then,*

i) $P \in A^k(M)$ *is* Δ*-homogeneous of degree* n *if and only if* $\{f_1, \ldots, f_k\}_P$ *is* Δ*-homogeneous of degree* $n + k$*, for all* $f_1, \ldots, f_k \in S^1_\Delta(M)$*, where* $\{\cdot, \ldots, \cdot\}_P$ *is the bracket defined as in (2.3).*

ii) $D \in D^k(M)$ *is* Δ*-homogeneous of degree* n *if and only if* $\{f_1, \ldots, f_k\}_D$ *is* Δ*-homogeneous of degree* $n + deg(f_1) + \ldots + deg(f_k)$*, for all* Δ*-homogeneous functions* $f_1, \ldots, f_k$ *of degree* 1 *or* 0.

Proof.- *i)* follows from the identity

$$\Delta(\{f_1, \ldots, f_k\}_P) = \langle [\![\Delta, P]\!]_M, df_1 \wedge \ldots \wedge df_k\rangle + \langle P, \mathcal{L}_\Delta(df_1 \wedge \ldots \wedge df_k)\rangle,$$

for $f_1, \ldots, f_k \in C^\infty(M)$, where $\mathcal{L}$ denotes the usual Lie derivative operator, and the fact that $df_1 \wedge \ldots \wedge df_k$, with Δ-linear functions $f_1, \ldots, f_k$, generate $\wedge^k T^*M$ over an open-dense subset.

The proof of *ii)* is analogous. □

Next, we will consider the particular case when Δ is the Liouville vector field Δ_E on a vector bundle E. We recall that, in such a case, $S^1_{\Delta_E}(E)$ is the space of linear functions on E and $S^0_{\Delta_E}(E)$ is the space of basic functions on E (see Example 3.4).

Corollary 3.7 *Let* $E \to M$ *be a vector bundle over* M*,* Δ_E *be the Liouville vector field on* E *and* (E, Δ_E) *be the corresponding strict homogeneous structure. Then:*

(i) $P \in A^k(E)$ *is* Δ_E*-homogeneous if and only if* P *is linear, that is,*

$$\{f_1, \ldots, f_k\}_P \in S^1_{\Delta_E}(E), \text{ for } f_1, \ldots, f_k \in S^1_{\Delta_E}(E). \tag{3.1}$$

(ii) $D \in D^k(M)$ *is* Δ_E*-homogeneous if and only if*

$$\begin{array}{ll} \{f_1, \dots, f_k\}_D \in S^1_{\Delta_E}(E), & \textit{for } f_1, \dots, f_k \in S^1_{\Delta_E}(E), \\ \{1, f_2, \dots, f_k\}_D \in S^0_{\Delta_E}(E), & \textit{for } f_2, \dots, f_k \in S^1_{\Delta_E}(E). \end{array} \tag{3.2}$$

Proof.- (i) follows from Proposition 3.6.

On the other hand, if $D \in D^k(M)$ is Δ_E-homogeneous then, using again Proposition 3.6, we deduce that (3.2) holds.

Conversely, suppose that (3.2) holds.

Then, if $f_1^0 \in S^0_{\Delta_E}(E)$ and $f_1^1, \dots, f_k^1 \in S^1_{\Delta_E}(E)$, we have that

$$S^1_{\Delta_E}(E) \ni \{f_1^0 f_1^1, f_2^1, \dots, f_k^1\}_D = f_1^0\{f_1^1, f_2^1, \dots, f_k^1\}_D + f_1^1\{f_1^0, f_2^1, \dots, f_k^1\}_D - f_1^0 f_1^1\{1, f_2^1, \dots, f_k^1\}_D.$$

This implies that

$$f_1^1\{f_1^0, f_2^1, \dots, f_k^1\}_D \in S^1_{\Delta_E}(E), \quad \forall f_1^1 \in S^1_{\Delta_E}(E).$$

Thus,

$$\{f_1^0, f_2^1, \dots, f_k^1\}_D \in S^0_{\Delta_E}(E), \text{ for } f_1^0 \in S^0_{\Delta_E}(E) \text{ and } f_2^1, \dots, f_k^1 \in S^1_{\Delta_E}(E). \tag{3.3}$$

Now, we will see that

$$\{1, f_2^0, f_3^1, \dots, f_k^1\}_D = 0, \text{ for } f_2^0 \in S^0_{\Delta_E}(E) \text{ and } f_3^1, \dots, f_k^1 \in S^1_{\Delta_E}(E). \tag{3.4}$$

If $f_2^1 \in S^1_{\Delta_E}(E)$, we obtain that

$$S^0_{\Delta_E}(E) \ni \{1, f_2^0 f_2^1, f_3^1, \dots, f_k^1\}_D = f_2^0\{1, f_2^1, f_3^1, \dots, f_k^1\}_D + f_2^1\{1, f_2^0, f_3^1, \dots, f_k^1\}_D.$$

Therefore, we deduce that

$$f_2^1\{1, f_2^0, f_3^1, \dots, f_k^1\}_D \in S^0_{\Delta_E}(E), \quad \forall f_2^1 \in S^1_{\Delta_E}(E),$$

and, consequently,

$$\{1, f_2^0, f_3^1, \dots, f_k^1\}_D = 0.$$

Next, we will prove that

$$\{f_1^0, f_2^0, f_3^1, \dots, f_k^1\}_D = 0, \text{ for } f_1^0, f_2^0 \in S^0_{\Delta_E}(E) \text{ and } f_3^1, \dots, f_k^1 \in S^1_{\Delta_E}(E). \tag{3.5}$$

If $f_2^1 \in S^1_{\Delta_E}(E)$ then, using (3.3) and (3.4), we have that

$$S^0_{\Delta_E}(E) \ni \{f_1^0, f_2^0 f_2^1, f_3^1, \dots, f_k^1\}_D = f_2^0\{f_1^0, f_2^1, f_3^1, \dots, f_k^1\}_D + f_2^1\{f_1^0, f_2^0, f_3^1, \dots, f_k^1\}_D.$$

This implies that

$$f_2^1\{f_1^0, f_2^0, f_3^1, \dots, f_k^1\}_D \in S^0_{\Delta_E}(E), \quad \forall f_2^1 \in S^1_{\Delta_E}(E),$$

and thus (3.5) holds.

Proceeding as above, we also may deduce that

$$\{f_1^0, \dots, f_r^0, f_{r+1}^1, \dots, f_k^1\}_D = 0,$$

for $f_1^0, \dots, f_r^0 \in S^0_{\Delta_E}(E)$ and $f_{r+1}^1, \dots, f_k^1 \in S^1_{\Delta_E}(E)$, with $2 \leq r \leq k$.

Therefore, D is Δ_E-homogeneous (see Proposition 3.6). □

Remark 3.8 We remark that Poisson (Jacobi) structures which are homogeneous with respect to the Liouville vector field of a vector bundle play an important role in the study of mechanical systems. Some examples of these structures are the following: the canonical symplectic structure on the cotangent bundle T^*M of a manifold M, the Lie-Poisson structure on the dual space of a real Lie algebra of finite dimension, and the canonical contact structure on the product manifold $T^*M \times \mathbb{R}$ (for more details, see [IM1]).

3.2 Poisson-Jacobi reductive structures

Definition 3.9 *A* Poisson-Jacobi (PJ) reductive structure *is a triple* (M, N, Δ)*, where* (M, Δ) *is a homogeneous structure and* N *is a 1-codimensional closed submanifold of* M *such that* Δ *is transversal to* N*.*

From Proposition 3.2, we deduce the following result.

Proposition 3.10 *Let* (M, N, Δ) *be a PJ reductive structure. Then, there is a tubular neighborhood* U *of* N *in* M *such that* $(U, N, \Delta_{|U})$ *is diffeomorphically equivalent to the free PJ reductive structure* $(N \times \mathbb{R}, N, \partial_s)$*.*

Now, we pass to the main result of the paper.

Let (M, N, Δ) be a PJ reductive structure. Let us consider a tubular neighborhood U of N, like in Proposition 3.8. There is the unique function $\tilde{1}_N \in S^1_\Delta(U)$ such that $(\tilde{1}_N)_{|N} \equiv 1$. Under the diffeomorphism between U and $N \times \mathbb{R}$, $\tilde{1}_N$ is the positive function on $N \times \mathbb{R}$

$$(x, s) \in N \times \mathbb{R} \to e^s \in \mathbb{R}.$$

Let us denote by $\mathcal{F}$ the foliation defined as the level sets of this function and by $A(\mathcal{F})$, $D(\mathcal{F})$ the spaces of elements of $A(U)$, $D(U)$ which are tangent to $\mathcal{F}$. Here we call $P \in A^k(U)$ *tangent to* $\mathcal{F}$ if $P_x \in \wedge^k T_x\mathcal{F}_x$, where $\mathcal{F}_x$ is the leaf of $\mathcal{F}$ containing $x \in U$. Consequently, $P^0 + I_U \wedge P^1 \in D^k(U)$ is tangent to $\mathcal{F}$ if $P^0 \in A^k(U)$ and $P^1 \in A^{k-1}(U)$ are tangent to $\mathcal{F}$.

It is obvious that any $P \in A^k(U)$ has a unique decomposition $P = P^0_{\mathcal{F}} + \Delta_{|U} \wedge P^1_{\mathcal{F}}$, where $P^0_{\mathcal{F}} \in A^k(\mathcal{F})$ and $P^1_{\mathcal{F}} \in A^{k-1}(\mathcal{F})$. We can use this decomposition to define, for each $P \in A^k(U)$, operators $J(P) \in D^k(U)$ and $J_N(P) \in D^k(N)$ by the formulae

$$J(P) = P^0_{\mathcal{F}} + I_U \wedge P^1_{\mathcal{F}}$$

and

$$J_N(P) = J(P)_{|N}.$$

Theorem 3.11 *Let* (M, N, Δ) *be a PJ reductive structure and let* U *be a tubular neighborhood of* N *in* M *as in Proposition 3.10. Then:*

i) The mapping J *defines a one-to-one correspondence between* $\Delta_{|U}$*-homogeneous multivector fields on* U *and* $\Delta_{|U}$*-homogeneous first-order polydifferential operators on* U *which are tangent to the foliation* $\mathcal{F}$*;*

ii) The mapping J_N defines a one-to-one correspondence between $\Delta_{|U}$-homogeneous multivector fields on U and first-order polydifferential operators on N.

Moreover,

(a) $\{f_1,\ldots,f_k\}_P = \{f_1,\ldots,f_k\}_{J(P)}$ *and* $(\{f_1,\ldots,f_k\}_P)_{|N} = \{f_{1|N},\ldots,f_{k|N}\}_{J_N(P)}$

(b) $[\![J(P),J(Q)]\!]^1_U = J([\![P,Q]\!]_U)$ *and* $[\![J_N(P),J_N(Q)]\!]^1_N = J_N([\![P,Q]\!]_U)$,

for all $f_1,\ldots,f_k \in S^1_\Delta(U)$ and $\Delta_{|U}$-homogeneous tensors $P,Q \in A(U)$.

Proof.- The tensors $J(P)$ and $J_N(P)$ clearly satisfy (a).

Note that the foliation $\mathcal{F}$ is Δ-invariant, since $\tilde{1}_N$ is Δ-homogeneous. This implies that $[\![\Delta, A(\mathcal{F})]\!]_M \subset A(\mathcal{F})$, so that $[\![\Delta_{|U}, P^0_{\mathcal{F}}]\!]_U + \Delta_{|U} \wedge [\![\Delta_{|U}, P^1_{\mathcal{F}}]\!]_U$ is the decomposition of $[\![\Delta_{|U}, P]\!]_U$ for each tensor $P = P^0_{\mathcal{F}} + \Delta_{|U} \wedge P^1_{\mathcal{F}} \in A^k(U)$. This means that if P is $\Delta_{|U}$-homogeneous then $J(P)$ is also $\Delta_{|U}$-homogeneous. Conversely, for a pair $P^0 \in A^k(\mathcal{F})$, $P^1 \in A^{k-1}(\mathcal{F})$, $\Delta_{|U}$-homogeneous of degree $1-k$, the operator $P = P^0 + \Delta_{|U} \wedge P^1$ is $\Delta_{|U}$-homogeneous. Thus, J is bijective.

Now, due to the fact that for homogeneous P, $[\![\Delta_{|U}, P]\!]_U = (1-k)P = [\![I_U, P]\!]^1_U$, we get by direct calculations using the properties of the Schouten-Jacobi bracket that (b) is satisfied.

To prove (ii) we notice first that for a $\Delta_{|U}$-homogeneous P, the operator $(\tilde{1}_N)^{k-1}J(P)$ is homogeneous of degree zero, i.e. it is $\Delta_{|U}$-invariant. It follows that $(\tilde{1}_N)^{k-1}J(P)$ and $J(P)$ are uniquely determined by $J_N(P)$. To show that J_N is surjective, let us take $D_N = P^0_N + I_N \wedge P^1_N \in D^k(N)$. There are unique $\bar{P}^0 \in A^k(U)$, $\bar{P}^1 \in A^{k-1}(U)$ which are $\Delta_{|U}$-invariant and equal to P^0_N and P^1_N, respectively, when restricted to N. We just use the flow of $\Delta_{|U}$ to extend tensors on N to $\Delta_{|U}$-invariant tensors on U. Then $\tilde{P}^0 = (\tilde{1}_N)^{1-k}\bar{P}^0$ and $\tilde{P}^1 = (\tilde{1}_N)^{1-k}\bar{P}^1$ give rise to a $\Delta_{|U}$-homogeneous tensor $\tilde{P} = \tilde{P}^0 + \Delta_{|U} \wedge \tilde{P}^1$, with $J_N(\tilde{P}) = D_N$. □

Remark 3.12 *i)* The above result is a generalization of the main theorem in [DLM] which states that Δ-homogeneous Poisson tensors on M can be reduced to Jacobi structures on N. Indeed if Λ is Poisson, then $[\![\Lambda,\Lambda]\!]_{|U} = 0$, so $[\![J_N(\Lambda), J_N(\Lambda)]\!]^1_N = 0$ which exactly means that $J_N(\Lambda)$ is a Jacobi structure on N (see [GM1, IM2]). Actually, it is a sort of a super-Poissonization. Indeed, the Nijenhuis-Schouten bracket $[\![\cdot,\cdot]\!]_M$ on M is a graded (or super) Poisson bracket, while the Schouten-Jacobi bracket $[\![\cdot,\cdot]\!]^1_M$ on N is a graded (or super) Jacobi bracket (cf. [GM2]).

ii) We call this construction a Poisson-Jacobi reduction, since it is a half way of the Poisson-Poisson reduction in the case when $\Gamma = i_{\phi_N} J_N(\Lambda)$ is the vector field on N whose orbits have a manifold structure. Then, the bracket $\{\cdot,\ldots,\cdot\}_{J_N(\Lambda)}$ restricted to functions which are constant on orbits of Γ gives a Poisson bracket on N/Γ. In the case when M is symplectic, the Poisson structure on N/Γ obtained in this way is the standard symplectic reduction of the Poisson structure associated with a symplectic form Ω on M with respect to the coisotropic submanifold N. An explicit example of the above construction is the following one. Suppose that the manifold M is $\mathbb{R}^{2n}$, the submanifold N is the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ and the vector field Δ on $\mathbb{R}^{2n}$ is

$$\Delta = \frac{1}{2}\sum_{i=1}^{n}(q^i\partial_{q^i} + p_i\partial_{p_i}),$$

where $(q^i, p_i)_{i=1,\dots,n}$ are the usual coordinates on $\mathbb{R}^{2n}$. It is clear that Δ is transversal to N. Actually, the map

$$\mathbb{R}^{2n} - \{0\} \to S^{2n-1} \times \mathbb{R}, \qquad x \to (\frac{x}{\|x\|}, \ln \|x\|^2)$$

is a diffeomorphism of $\mathbb{R}^{2n} - \{0\}$ onto $S^{2n-1} \times \mathbb{R} = N \times \mathbb{R}$ which maps $\Delta_{|\mathbb{R}^{2n}-\{0\}}$ into ∂_s. Thus, we will take as a tubular neighborhood of $N = S^{2n-1}$ in $M = \mathbb{R}^{2n}$ the open subset $U = \mathbb{R}^{2n} - \{0\}$. Now, let Λ be the 2-vector on M defined by

$$\Lambda = \sum_{i=1}^{n} (\partial_{q^i} \wedge \partial_{p_i}).$$

Λ is the Poisson structure associated with the canonical symplectic 2-form ω on $M = \mathbb{R}^{2n}$ given by $\omega = \displaystyle\sum_{i=1}^{n} dq^i \wedge dp_i$.

A direct computation proves that $\Lambda_{|U}$ is a $\Delta_{|U}$-homogeneous Poisson structure. Therefore, it induces a Jacobi structure $J_N(\Lambda_{|U})$ on $N = S^{2n-1}$. Note that $J_N(\Lambda_{|U})$ is just the Jacobi structure associated with the canonical contact 1-form η on S^{2n-1} defined by

$$\eta = \frac{1}{2} j^*(\sum_{i=1}^{n} (q^i dp_i - p_i dq^i)),$$

where $j : S^{2n-1} \to \mathbb{R}^{2n}$ is the canonical inclusion (for the definition of the Jacobi structure associated with a contact 1-form, see, for instance, [ChLM]). This Poisson-Jacobi reduction can be associated also with a reduction with respect to a Hamiltonian action of S^1 on $\mathbb{R}^{2n}$. Indeed, consider the harmonic oscillator Hamiltonian $H : \mathbb{R}^{2n} \to \mathbb{R}$ given by

$$H = \frac{1}{2} \sum_{i=1}^{n} ((q^i)^2 + (p_i)^2)$$

and the Hamiltonian vector field $\mathcal{H}_H^\Lambda = i_{dH}(\Lambda)$ of H with respect to Λ, that is,

$$\mathcal{H}_H^\Lambda = \sum_{i=1}^{n} (p_i \partial_{q^i} - q^i \partial_{p_i}).$$

The orbit of $\mathcal{H}_H^\Lambda$ passing through (q^i, p_i) is the curve $\alpha_{(q^i,p_i)} : \mathbb{R} \to \mathbb{R}^{2n}$ on $\mathbb{R}^{2n}$

$$\begin{aligned} \alpha_{(q^i,p_i)}(t) \quad = \quad & (q^1 \cos t + p_1 \sin t, \dots, q^n \cos t + p_n \sin t, \\ & p_1 \cos t - q^1 \sin t, \dots, p_n \cos t - q^n \sin t). \end{aligned}$$

Consequently, $\alpha_{(q^i,p_i)}$ is periodic with period 2π which implies that the flow of $\mathcal{H}_H^\Lambda$ defines a symplectic action of S^1 on $\mathbb{R}^{2n}$ with the momentum map given by H. Moreover, the restriction Γ of $\mathcal{H}_H^\Lambda$ to S^{2n-1} is tangent to S^{2n-1} and Γ is a regular vector field on S^{2n-1}, that is, the space of orbits of Γ, S^{2n-1}/Γ, has a manifold structure and, thus, $S^{2n-1}/\Gamma \cong S^{2n-1}/S^1$ is a symplectic manifold. Actually, the reduced symplectic space S^{2n-1}/S^1 is the complex projective space with the standard symplectic structure.

iii) We call the inverse of the map $P \mapsto J_N(P) = D_N$ the *Poissonization* of $D_N \in D^k(N)$. This map is a homomorphism of the Schouten-Jacobi bracket on $D(N)$ into the Schouten-Nijenhuis bracket of Δ-homogeneous multivector fields in a neighborhood of N in M. In particular, it maps Jacobi structures into Poisson structures. For free PJ reductive structures we get, like in [DLM] for the case $k = 2$, that the Poissonization of $D_N = P_N^0 + I_N \wedge P_N^1$ is $\mathrm{e}^{(1-k)s}(P_N^0 + \partial_s \wedge P_N^1)$ on $N \times \mathbb{R}$.

Using Theorem 3.11 and generalizing Remark 3.12 *i)*, we have the following result which relates homogeneous Nambu-Poisson tensors on M to Nambu-Jacobi tensors on N (see [MVV, T] for the definition of a Nambu-Poisson and a Nambu-Jacobi tensor).

Corollary 3.13 *Let (M, N, Δ) be a PJ reductive structure. For a tubular neighborhood U of N in M there is a one-to-one correspondence between $\Delta_{|U}$-homogeneous Nambu-Poisson tensors on M and Nambu-Jacobi tensors on N.*

Proof.- We know that a tensor $P \in A^k(M)$ on a manifold M is Nambu-Poisson if and only if

$$[\![[\![\dots[\![[\![P, f_1]\!]_M, f_2]\!]_M, \dots, f_{k-1}]\!]_M, P]\!]_M = 0, \tag{3.6}$$

for $f_1, \dots f_{k-1} \in C^\infty(M)$ and that $D \in D^k(M)$ is a Nambu-Jacobi structure on M if and only if

$$[\![[\![\dots[\![[\![D, f_1]\!]^1_M, f_2]\!]^1_M, \dots, f_{k-1}]\!]^1_M, D]\!]^1_M = 0, \tag{3.7}$$

for $f_1, \dots f_{k-1} \in C^\infty(M)$.

Therefore, our result follows from (3.6), (3.7) and Theorem 3.11. $\square$

The above result is local. We can get global results in particular classes. The following one has been proved in [GIMPU] for bivector fields by a different method.

Theorem 3.14 *Let $E \to M$ be a vector bundle of rank n, $n > 1$, and let A be an affine hyperbundle of E, i.e. an affine subbundle of rank $(n-1)$ and not intersecting the 0-section of E. Then, the association $P \mapsto J_A(P)$ establishes a one-to-one correspondence between Δ_E-homogeneous tensors $P \in A^k(E)$, the vector field Δ_E being the Liouville vector field, and those $D_A \in D^k(A)$ which are affine, i.e. such that $\{h_1, \dots, h_k\}_{D_A}$ is affine whenever $h_1, \dots, h_k$ are affine (along fibers) functions on A. Moreover, for this correspondence,*

$$[\![J_A(P), J_A(Q)]\!]^1_A = J_A([\![P, Q]\!]_E). \tag{3.8}$$

Proof of Theorem 3.14.- The Liouville vector field Δ_E is clearly transversal to A, so the association $P \mapsto J_A(P)$ satisfies

$$(\{f_1, \dots, f_k\}_P)_{|A} = \{f_{1|A}, \dots, f_{k|A}\}_{J_A(P)}$$

and (3.8) according to Theorem 3.11. The affine functions on A are exactly restrictions of linear functions on E (see the next Lemma 3.15), so $J_A(P)$ is affine.

Conversely, according to Theorem 3.11, there is a neighbourhood U of A in E on which Δ_E nowhere vanishes and a $(\Delta_E)_{|U}$-homogeneous k-vector field P_U on U such that $D_A = J_A(P_U)$. We will show that P_U is *linear*, i.e. that $\{(f_1)_{|U}, \dots, (f_k)_{|U}\}_{P_U}$ is the restriction to U of a linear function on E for all linear functions $f_1, \dots, f_k$ on E. In the case of a 0-tensor, i.e. a function $f \in C^\infty(U)$, this means that f is the restriction to U of a linear function on E.

Indeed, since by Theorem 3.11

$$(\{(f_1)_{|U}, \dots, (f_k)_{|U}\}_{P_U})_{|A} = \{f_{1|A}, \dots, f_{k|A}\}_{D_A},$$

the function $\{(f_1)_{|U}, \dots, (f_k)_{|U}\}_{P_U}$ is Δ_E-homogeneous on U and its restriction to A is affine, thus it is the restriction to U of a linear function. Note that every affine function on A has a unique extension to a linear function on the whole E (see the next Lemma 3.15). Moreover, two Δ_E-homogeneous functions

f and g on U which coincide on A must coincide on the Δ_E orbits of points from A and , since A is an affine hyperbundle of E not intersecting the 0-section of E, we deduce that $f = g$ on U.

What remains to be proven is that P_U has a unique extension to a Δ_E-homogeneous tensor on E that follows from the next Lemma 3.16. □

Lemma 3.15 *Let E be a real vector bundle over M and A be an affine hyperbundle of E not intersecting the 0-section $0 : M \to E$ of E. Suppose that A^+ is the real vector bundle over M whose fiber at the point $x \in M$ is the real vector space $A^+_x = Aff(A_x, \mathbb{R})$, that is, A^+_x is the space of real affine functions on A_x. Then, the map $R_A : E^* \to A^+$ defined by $R_A(\alpha_x) = (\alpha_x)_{|A_x}$, for $\alpha_x \in E^*_x$ is an isomorphism of vector bundles.*

Proof.- Let x be a point of M and $\alpha_x \in E^*_x$. Then, it is easy to prove that $R_A(\alpha_x) \in A^+_x$ and that the map $(R_A)_{|E^*_x} : E^*_x \to A^+_x$ is linear. Moreover, if $R_A(\alpha_x) = 0$, we have that $(\alpha_x)_{|A_x} = 0$ and, using that $0(x) \notin A_x$, we conclude that $\alpha_x = 0$. Thus, $(R_A)_{|E^*_x}$ is injective and, since $\dim E^*_x = \dim A^+_x = n$, we conclude that $(R_A)_{|E^*_x} : E^*_x \to A^+_x$ is a linear isomorphism. This proves the result. □

Lemma 3.16 *Let $\tau : E \to M$ be a vector bundle of rank n, $n >1$, A be an affine hyperbundle of E not intersecting the 0-section of E and U be a neighborhood of A in E. If P is a linear-homogeneous k-contravariant tensor field on U then P has a unique extension to a Δ_E-homogeneous (linear) k-contravariant tensor field $\tilde{P}$ on E.*

Proof.- The statement is local in M, so let us choose local coordinates $x = (x^a)$ in $V \subset M$ and the adapted linear coordinates (x^a, ξ_i) on $E_{|V}$, associated with a choice of a basis of local sections of $E_{|V}$. In these coordinates, the tensor P can be written in the form

$$\begin{aligned} P = {} & \textstyle\sum_{i_1,\dots,i_k} f^k_{\xi_{i_1},\dots,\xi_{i_k}}(x,\xi)\partial_{\xi_{i_1}} \otimes \cdots \otimes \partial_{\xi_{i_k}} + \\ & + \textstyle\sum_{i_1,\dots,i_{k-1},a} f^{k-1}_{\xi_{i_1},\dots,\xi_{i_{k-1}},x^a}(x,\xi)\partial_{\xi_{i_1}} \otimes \cdots \otimes \partial_{\xi_{i_{k-1}}} \otimes \partial_{x^a} + \\ & + \textstyle\sum_{i_1,\dots,i_{k-1},a} f^{k-1}_{\xi_{i_1},\dots,\xi_{i_{k-2}},x^a,\xi_{i_{k-1}}}(x,\xi)\partial_{\xi_{i_1}} \otimes \cdots \otimes \partial_{\xi_{k-2}} \otimes \partial_{x^a} \otimes \partial_{\xi_{i_{k-1}}} + \cdots + \\ & + \textstyle\sum_{a_1,\dots,a_k} f^0_{x^{a_1},\dots,x^{a_k}}(x,\xi)\partial_{x^{a_1}} \otimes \cdots \otimes \partial_{x^{a_k}}. \end{aligned} \tag{3.9}$$

By linearity of the tensor P, $\{\xi_{i_1}, \dots, \xi_{i_k}\}_P = f^k_{\xi_{i_1},\dots,\xi_{i_k}}(x,\xi)$ is linear in ξ, so it can be extended uniquely to a linear function on the whole $E_{|V}$. Similarly, proceeding by induction with respect to m one can show that the linearity of

$$\{\xi_{i_1}, \dots, x^{a_1} \cdot \xi_{j_1}, \dots, x^{a_m} \cdot \xi_{j_m}, \dots, \xi_{i_{k-m}}\}_P$$

implies that

$$f^{k-m}_{\xi_{i_1},\dots,x^{a_1},\dots,x^{a_m},\dots,\xi_{i_{k-m}}}(x,\xi) \cdot \xi_{j_1} \cdots \xi_{j_m} \tag{3.10}$$

is linear for all $j_1, \dots, j_m$. Once we know that (3.10) are linear, it is easy to see that

$$f^{k-1}_{\xi_{i_1},\dots,x^{a_1},\dots,\xi_{i_{k-1}}}(x,\xi) \tag{3.11}$$

is constant on fibers, so it extends uniquely to a function which is constant on the fibers of $E_{|V}$. On the other hand, since $n > 1$ and U is a neighborhood of A in E, there exist $i_1, \dots, i_{n-1} \in \{1, \dots, n\}$ such that

$$U \cap \{\xi_{i_k} = 0\} \neq \emptyset, \quad \forall k \in \{1, \dots, n-1\}.$$

Using this fact and the linearity of (3.10), we deduce that

$$f^{k-m}_{\xi_{i_1},\ldots,x^{a_1},\ldots,x^{a_m},\ldots,\xi_{i_{k-m}}}(x,\xi)=0, \quad \text{for } m>1.$$

Note that if $rank(E)=1$, we have that $\xi_{i_l}=\xi$ and there is another possibility, namely

$$f^{k-m}_{\xi_{i_1},\ldots,x^{a_1},\ldots,x^{a_m},\ldots,\xi_{i_{k-m}}}(x,\xi)=g(x)\xi^{1-m},$$

which clearly does not prolong onto $E_{|V}$ analytically along fibers. Now we define the prolongation $\tilde{P}_V$ of P to $E_{|V}$ by the formula (3.9) but with the prolonged coefficients. It is obvious that this constructed prolongation $\tilde{P}_V$ of P to $E_{|V}$ is homogeneous. By uniqueness of this homogeneous prolongation on every $E_{|V}$ for V running through an open covering of M, we get a unique homogeneous prolongation to the whole E. $\square$

Remark 3.17 The linearity cannot be replaced by Δ_E-homogeneity in the above lemma. The simplest counterexample is just the function $f(x)=|x|$ which is $x\partial_x$-homogeneous on $U=\mathbb{R}\setminus\{0\}$ but it is not linear on U.

Finally, we will prove a dual version of Theorem 3.11.

Let (M,N,Δ) be a PJ reductive structure and let U be a tubular neighborhood of N in M as in Proposition 3.10. The space of sections of the vector bundle $\wedge^k(T^1U)^*\to U$ (respectively, $\wedge^k(T^1N)^*\to N$) is $\Omega^k(U)\oplus\Omega^{k-1}(U)$ (respectively, $\Omega^k(N)\oplus\Omega^{k-1}(N)$) and it is obvious that any $\alpha\in\Omega^k(U)$ has a unique decomposition

$$\alpha=\tilde{1}_N(\alpha^0+d(\ln\tilde{1}_N)\wedge\alpha^1), \tag{3.12}$$

where $(\alpha^0,\alpha^1)\in\Omega^k(U)\oplus\Omega^{k-1}(U)$ and

$$i_{\Delta_{|U}}\alpha^0=0, \quad i_{\Delta_{|U}}\alpha^1=0.$$

Indeed, since $i_{\Delta_{|U}}d(\ln\tilde{1}_N)=1$, the form α^1 is defined by $\alpha^1=(\tilde{1}_N)^{-1}i_{\Delta_{|U}}\alpha$ and $\alpha^0=(\tilde{1}_N)^{-1}\alpha-d(\ln\tilde{1}_N)\wedge\alpha^1$. We can use this decomposition to define, for each $\alpha\in\Omega^k(U)$, a section $\Psi(\alpha)$ of the vector bundle $\wedge^k(T^1U)^*\to U$ by the formula

$$\Psi(\alpha)=(\alpha^0,\alpha^1).$$

On the other hand, a section $(\alpha^0,\alpha^1)\in\Omega^k(U)\oplus\Omega^{k-1}(U)$ is said to be $\Delta_{|U}$-basic if α^0 and α^1 are basic forms with respect to $\Delta_{|U}$, that is,

$$i_{\Delta_{|U}}\alpha^0=0, \quad i_{\Delta_{|U}}\alpha^1=0, \quad \mathcal{L}_{\Delta_{|U}}\alpha^0=0, \quad \mathcal{L}_{\Delta_{|U}}\alpha^1=0.$$

In addition, we will denote by $j:N\to U$ the canonical inclusion and by $\Psi_N:\Omega^k(U)\to\Omega^k(N)\oplus\Omega^{k-1}(N)$ the map defined by

$$\Psi_N(\alpha)=(\alpha^0_N,\alpha^1_N), \quad \text{for } \alpha\in\Omega^k(U),$$

where $\alpha^0_N=j^*(\alpha)$, $\alpha^1_N=j^*(i_{\Delta_{|U}}\alpha)$. On the other hand, from (3.12), it follows that

$$j^*\alpha=j^*\alpha^0, \quad j^*(i_{\Delta_{|U}}\alpha)=j^*\alpha^1, \tag{3.13}$$

(note that $j^*(\tilde{1}_N)$ is the constant function 1 on N), so $\alpha^0_N=j^*(\alpha^0)$ and $\alpha^1_N=j^*(\alpha^1)$.

Theorem 3.18 *Let (M, N, Δ) be a PJ reductive structure and let U be a tubular neighborhood of N in M as in Proposition 3.10. Then:*

(i) The map Ψ defines a one-to-one correspondence between the space of k-forms on U which are Δ_U-homogeneous of degree 1 and the space of sections of the vector bundle $\wedge^k(T^1U)^* \to U$ which are $\Delta_{|U}$-basic.

(ii) The map Ψ_N defines a one-to-one correspondence between the space of k-forms on U which are $\Delta_{|U}$-homogeneous of degree 1 and the space of sections of the vector bundle $\wedge^k(T^1N)^* \to N$, that is, $\Omega^k(N) \oplus \Omega^{k-1}(N)$.

Moreover, if $\alpha \in \Omega^k(U)$ is $\Delta_{|U}$-homogeneous of degree 1 then

$$\Psi(d_U\alpha) = d^1_U(\Psi\alpha), \quad \Psi_N(d_U\alpha) = d^1_N(\Psi_N\alpha),$$

where d_U is the usual exterior differential on U and d^1_U (respectively, d^1_N) is the Jacobi differential on U (respectively, N).

Proof.- Let α be a k-form on U,

$$\alpha = \tilde{1}_N(\alpha^0 + d(\ln \tilde{1}_N) \wedge \alpha^1), \tag{3.14}$$

with $(\alpha^0, \alpha^1) \in \Omega^k(U) \oplus \Omega^{k-1}(U)$ satisfying $i_{\Delta_{|U}}\alpha^0 = 0$ and $i_{\Delta_{|U}}\alpha^1 = 0$. Then

$$\mathcal{L}_{\Delta_{|U}}\alpha = \alpha + \tilde{1}_N(\mathcal{L}_{\Delta_{|U}}\alpha^0 + d(\ln \tilde{1}_N) \wedge \mathcal{L}_{\Delta_{|U}}\alpha^1).$$

Thus, since $i_{\Delta_{|U}}(\mathcal{L}_{\Delta_{|U}}\alpha^0) = 0$ and $i_{\Delta_{|U}}(\mathcal{L}_{\Delta_{|U}}\alpha^1) = 0$, we conclude that α is $\Delta_{|U}$-homogeneous of degree 1 if and only if α^0 and α^1 are $\Delta_{|U}$-basic. This proves (i).

Since

$$j^*\alpha = j^*\alpha^0, \quad j^*(i_{\Delta_{|U}}\alpha) = j^*\alpha^1,$$

using (i) and the fact that the map $j^* : \Omega^r(U) \to \Omega^r(N)$ defines a one-to-one correspondence between the space of $\Delta_{|U}$-basic r-forms on U and $\Omega^r(N)$, we deduce (ii).

Finally, if $\alpha \in \Omega^k(U)$ is $\Delta_{|U}$-homogeneous of degree 1 then, from (3.12), we obtain that

$$d_U\alpha = \tilde{1}_N(d_U\alpha^0 + d_U(\ln \tilde{1}_N) \wedge (\alpha^0 - d_U\alpha^1))$$

and, since

$$i_{\Delta_{|U}}(d_U\alpha^0) = \mathcal{L}_{\Delta_{|U}}\alpha^0 = 0, \quad i_{\Delta_{|U}}(\alpha^0 - d_U\alpha^1) = -\mathcal{L}_{\Delta_{|U}}\alpha^1 = 0,$$

we conclude that (see (3.13))

$$\Psi(d_U\alpha) = (d_U\alpha^0, \alpha^0 - d_U\alpha^1) = d^1_U(\Psi\alpha),$$

$$\Psi_N(d_U\alpha) = (d_N(j^*(\alpha^0)), j^*(\alpha^0) - d_N(j^*(\alpha^1))) = d^1_N(\Psi_N\alpha).$$

□

Using Theorem 3.18, one may recover the following well-known result (see, for instance, [MS, Proposition 3.58]).

Corollary 3.19 *If ω is a $\Delta_{|U}$-homogeneous of degree 1 symplectic form on U, then $\eta = \omega_N^1$ is a contact form on N. The Jacobi structure associated with η is $J_N(\Lambda)$, where Λ is the $\Delta_{|U}$-homogeneous Poisson tensor associated with ω.*

Proof.- Since, according to Theorem 3.18,

$$0 = \Psi_N(d\omega) = d_N^1(\Psi_N\omega) = (d\omega_N^0, \omega_N^0 - d\omega_N^1),$$

we have

$$d\eta = d\omega_N^1 = \omega_N^0 = j^*\omega. \tag{3.15}$$

If the dimension of N is $2k+1$ then (3.15) implies

$$(d\eta)^{2k} \wedge \eta = j^*(\omega^{2k} \wedge i_{\Delta_{|U}}\omega) = \frac{1}{k+1} j^*(i_{\Delta_{|U}}\omega^{2(k+1)}).$$

But $\omega^{2(k+1)} \neq 0$ on U (the form ω is symplectic) and Δ is transversal to N, so $j^*(i_{\Delta_{|U}}\omega^{2(k+1)}) \neq 0$, thus $(d\eta)^{2k} \wedge \eta \neq 0$ on N and, therefore, η is a contact 1-form on N. The contact form η induces an isomorphism of vector bundles $\flat_\eta : TN \to T^*N$ which on sections takes the form

$$\flat_\eta(X) = \langle \eta, X\rangle\eta - i_X d\eta. \tag{3.16}$$

The Jacobi bracket $\{f,g\}_\eta$ induced by η is given by $\{f,g\}_\eta = \mathcal{H}_f^\eta(g) - g\Gamma(f)$, where $\mathcal{H}_f^\eta$ is the 'Hamiltonian vector field' of $f \in C^\infty(N)$ defined by

$$\flat_\eta(\mathcal{H}_f^\eta) = (df - \Gamma(f)\eta) + f\eta$$

and Γ is the Reeb vector field of η determined by $\flat_\eta(\Gamma) = \eta$, i.e. $i_\Gamma d\eta = 0$ and $\langle\eta, \Gamma\rangle = 1$. Let $\{\cdot,\cdot\}_\omega$ be the Poisson bracket induced by the symplectic form ω. Due to Theorem 3.11, it remains to prove that $\{f,g\}_\eta = (\{\widetilde{f},\widetilde{g}\}_\omega)_{|N}$, where $\widetilde{f}$ denotes the unique extension of $f \in C^\infty(N)$ to a $\Delta_{|U}$-homogeneous function on U. Denote by $\mathcal{H}_{\tilde{f}}^\omega$ the Hamiltonian vector field of $\widetilde{f}$ with respect to ω, i.e. $-i_{\mathcal{H}_{\tilde{f}}^\omega}\omega = d\widetilde{f}$. It is easy to see that the Reeb vector field of η is $\widetilde{\Gamma}_{|N}$, $\widetilde{\Gamma} = \mathcal{H}_{\tilde{1}_N}^\omega$, and that $\mathcal{H}_f^\eta = (\mathcal{H}_{\tilde{f}}^\omega + \widetilde{\Gamma}(\widetilde{f})\Delta_{|U})_{|N}$, i.e. $\mathcal{H}_f^\eta$ is the projection of $\mathcal{H}_{\tilde{f}}^\omega$ along Δ onto N. We have

$$\{f,g\}_\eta = \mathcal{H}_f^\eta(g) - g\Gamma(f) = ((\mathcal{H}_{\tilde{f}}^\omega + \widetilde{\Gamma}(\widetilde{f})\Delta_{|U})(\widetilde{g}))_{|N} - g\Gamma(f).$$

Since $\Delta_{|U}(\widetilde{g}) = \widetilde{g}$, it follows that

$$\{f,g\}_\eta = (\mathcal{H}_{\tilde{f}}^\omega(\widetilde{g}))_{|N} = (\{\widetilde{f},\widetilde{g}\}_\omega)_{|N}.$$

$\square$

Remark 3.20 If $M = \mathbb{R}^{2n}$, Δ is the vector field on M defined by $\Delta = \frac{1}{2}\sum_{i=1}^n (q^i\partial_{q^i} + p_i\partial_{p_i})$, U is the open subset of M given by $U = \mathbb{R}^{2n} - \{0\}$, $\omega = \sum_{i=1}^n (dq^i \wedge dp_i)$ is the canonical $\Delta_{|U}$-homogeneous symplectic 2-form on U and N is the unit sphere S^{2n-1} in $\mathbb{R}^{2n}$ then η is the canonical contact 1-form on S^{2n-1} (see Remark 3.12, *ii*)).

Remark 3.21 A Poisson structure is a particular Lie algebra structure. A useful generalization of the latter in the graded case is a *(strongly) homotopy Lie algebra* (sh Lie algebra, L_∞-algebra) which appeared in the works of J. Stasheff and his collaborators [LM, LS]. Very close algebraic structures arose in physics as *string products* of B. Zwiebach [Zw]. An algebraic background of a homotopy Lie algebra

on a graded vector space V is a graded Lie algebra structure on the graded space $L(V) = \bigoplus_{n\geq 0} L^n(V)$ of (skew-symmetric) multilinear maps from V into V. The corresponding graded Lie bracket on $L(V)$ is actually a graded variant of the Nijenhuis-Richardson bracket $[\![\cdot,\cdot]\!]^{NR}$ and the homotopy Lie algebra on V is a formal series $B = \sum_{n\geq 0} B_n h^n$, $B_n \in L^n(V)$ with coefficients which satisfies the 'Master Equation' $[\![B, B]\!]^{NR} = 0$. One requires additionally that the degree of B_n is $n-2$. Of course, when B reduces to B_2, i.e. $B_n = 0$ for $n \neq 2$, we deal with a standard graded Lie bracket on V induced by $B_2 : V \times V \to V$ of degree 0. When also B_1 is non-trivial, the Jacobi identity for B_2 is satisfied only 'up to homotopy'. One can consider this general scheme skipping the assumption on the degree and one can work with any subalgebra of $L(V)$, also for non-graded V: we just consider the series B with coefficients in the Lie subalgebra of $L(V)$ and satisfying the Master Equation. Of course, this general scheme has nothing to do with 'homotopy' in general, when no grading on V or not proper degree of B_n is assumed.

In our case of the Schouten-Nijenhuis and Schouten-Jacobi brackets, one can consider their homotopy generalizations which respect the homogeneity, like these brackets do, and obtain the corresponding Poisson-Jacobi reduction on the level of homotopy algebras, but the detailed discussion of these problems exceeds limits of this note and we postpone it to a separate paper.

What we can have for free is the above scheme for the non-graded case of $V = C^\infty(M)$. The spaces $A^k(M)$ and $D^k(M)$ can be interpreted as subspaces of $L^n(V)$ and the brackets $[\![\cdot,\cdot]\!]_M$ and $[\![\cdot,\cdot]\!]^1_M$ are restrictions of $[\![\cdot,\cdot]\!]^{NR}$ to $A^k(M)$ and $D^k(M)$, respectively. A *formal Poisson* structure on M is a formal series $B = \sum_{n\geq 0} B_n h^n$, $B_n \in A^n(M)$ such that $[\![B, B]\!]_M = 0$, where we use the obvious extension of the Schouten-Nijenhuis bracket to formal series of multivector fields: $[\![B, B]\!]_M = \sum_{i,j} [\![B_i, B_j]\!]_M h^{i+j-1}$. By properties of the Schouten-Nijenhuis bracket, only the even part of B is relevant. If B_2 is the only non-trivial part of B, we recognize a standard Poisson structure. If this is the case of B_{2k}, we recognize a *generalized Poisson structure* in the sense of Azcárraga, Perelomov, and Pérez Bueno [APP1, APP2] (see also [AIP]). Now, according to Theorem 3.11, if B is Δ-homogeneous, we can reduce B to a *formal Jacobi* structure on the submanifold N by $J_N(B) = \sum_{i\geq 0} J_N(B_N)$, since

$$[\![J_N(B), J_N(B)]\!]^1_M = J_N([\![B, B]\!]_M) = 0.$$

In particular, this reduces generalized Poisson structures on M to *generalized Jacobi structures* on N, defined in obvious way (see [P]). Note also that the corresponding operators $\partial_B = ad_B$ and $\partial_{J_N(B)} = ad_{J_N(B)}$ act as 'homotopy differentials' in the graded Lie algebras $A^k(M)[[h]]$ and $D^k(M)[[h]]$, i.e. $\partial_B^2 = 0$ and $\partial^2_{J_N(B)} = 0$, generalizing the standard Poisson and Jacobi cohomology.

References

[AKSZ] M. Alexandrov; M. Kontsevich, A. Schwarz, O. Zaboronsky: The geometry of the master equation and topological quantum field theory, *Internat. J. Modern Phys. A* **12** (1997), no. 7, 1405–1429.

[AIP] J. A. Azcárraga, J. M. Izquierdo, J. C. Pérez Bueno: *On the higher-order generalizations of Poisson structures*, J. Phys. A: Math. Gen. **30** (1997), L607–L616.

[APP1] J. A. Azcárraga, A. M. Perelomov, J. C. Pérez Bueno: *New generalized Poisson structures*, J. Phys. A: Math. Gen. **29** (1996), L151–L157.

[APP2] J. A. Azcárraga, A. M. Perelomov, J. C. Pérez Bueno: *The Schouten-Nijenhuis brackets, cohomology, and generalized Poisson structures*, J. Phys. A: Math. Gen. **29** (1996), 7993–8009.

[ChLM] D. Chinea, M. de León, J.C. Marrero: Prequantizable Poisson manifolds and Jacobi structures, *J. Phys. A: Math. Gen.*, **29**, (1996), 6313-6324.

[DLM] P. Dazord, A. Lichnerowicz, Ch.M. Marle: Structure locale des variétés de Jacobi, *J. Math. Pures et Appl.*, **70** (1991), 101-152.

[Ge] E. Getzler: Batalin-Vilkovisky algebras and two-dimensional topological field theories, *Comm. Math. Phys.* **159** (1994), no. 2, 265–285.

[GIMPU] J. Grabowski, D. Iglesias, J.C. Marrero, E. Padrón, P. Urbanski: Affine Jacobi structures on vector and affine bundles, *Preprint (2002)*, *arXiv: math.DG/0212052*.

[GM1] J. Grabowski, G. Marmo: Jacobi structures revisited, *J. Phys. A: Math. Gen.*, **34** (2001), 10975-10990.

[GM2] J. Grabowski, G. Marmo: The graded Jacobi algebras and (co)homology, *J. Phys. A: Math. Gen.*, **36** (2003), 161-181.

[IM1] D. Iglesias, J.C. Marrero: Some linear Jacobi structures on vector bundles, *C.R. Acad. Sci. Paris*, **331** Sér. I (2000), 125-130.

[IM2] D. Iglesias, J.C. Marrero: Generalized Lie bialgebroids and Jacobi structures, *J. Geom. Phys.*, **40** (2001), 176-199.

[KoS] Y. Kosmann-Schwarzbach: Exact Gerstenhaber algebras and Lie bialgebroids, *Acta Appl. Math.*, **41** (1995), 153–165.

[KS2] Y. Kosmann-Schwarzbach: From Poisson algebras to Gerstenhaber algebras, *Ann. Inst. Fourier (Grenoble)* **46** (1996), no. 5, 1243–1274.

[LM] T. Lada, M. Markl: Strongly homotopy Lie algebras, *Comm. Algebra* **23** (1995), 2147–2161.

[LS] T. Lada, J. Stasheff: Introduction to sh Lie algebras for physicists, *Int. J. Theor. Phys.* **32** (1993), 1087–1103.

[Li] P. Libermann: On Liouville forms, in: *Poisson Geometry*, J. Grabowski and P. Urbański, eds., Banach Center Publications **51**, Warsaw 2000, 151–164.

[L1] A. Lichnerowicz: Les variétés de Poisson et leurs algébres de Lie associées, *J. Differential Geometry*, **12** (1977), 253-300.

[L2] A. Lichnerowicz: Les variétés de Jacobi et leurs algébres de Lie associées, *J. Math. pures et appl.*, **57** (1978), 453-488.

[M] K. Mackenzie: *Lie groupoids and Lie algebroids in differential geometry*, Cambridge University Press, 1987.

[MVV] G. Marmo, G. Vilasi, A.M. Vinogradov: The local structure of n-Poisson and n-Jacobi manifolds, *J. Geom. Phys.*, **25** (1998), 141-182.

[MS] D. McDuff, D. Salamon: Introduction to symplectic topology. Second edition. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 1998.

[N] Ngô-van-Quê: Sur l'espace de prolongement différentiable, *J. Diff. Geom.*, **2** (1968), 33-40.

[Ni] A. Nijenhuis: Jacobi-type identities for bilinear differential concomitants of certain tensor fields I., *Indag. Math.*, **17** (1995), 390-403.

[P] J. C. Pérez Bueno: *Generalized Jacobi structures*, J. Phys. A: Math. Gen. **30** (1997), 6509-6515.

[Sc] J. A. Schouten: Uber Differentialkonkomitanten zweier kontravarianter Größen, *Indag. Math.*, **2** (1940), 449-452.

[St] J. Stasheff: Closed string field theory, strong homotopy Lie algebras and the operad actions of moduli spaces, *Perspectives in mathematical physics*, 265–288, Conf. Proc. Lecture Notes Math. Phys., III, Internat. Press, Cambridge, MA, 1994.

[T] L. Takhtajan: On foundations of generalized Nambu mechanics, *Commun. Math. Phys.*, **160** (1994), 295-315.

[TU] W. Tulczyjew and P. Urbański: Differential forms on vector bundles, *Rep. Math. Phys.*, **45** (2000), 357-370.

[Wi] E. Witten: Supersymmetry and Morse theory, *J. Differential Geom.* **17** (1982), no. 4, 661–692 (1983).

[Zw] B. Zwiebach: Closed string field theory: quantum action and the Batalin-Vilkovisky master equation, *Nuclear Phys. B* **390** (1993), no. 1, 33–152.